arXiv:math/0405205v1 [math.DG] 12 May 2004

DIMENSIONAL REDUCTION OF
THE PERTURBED HERMITIAN–EINSTEIN EQUATION

STEVEN B. BRADLOW[1], JAMES F. GLAZEBROOK, AND FRANZ W. KAMBER[2]

ABSTRACT. Given a Kählerian holomorphic fiber bundle $F \hookrightarrow M \to X$, whose fiber F is a compact homogeneous Kähler manifold, we describe the perturbed Hermitian–Einstein equations relative to certain holomorphic vector bundles $\mathcal{E} \to M$. With respect to special metrics on $\mathcal{E}$, there is a dimensional reduction procedure which reduces this equation to a system of equations on X known as the twisted coupled vortex equations.

1. INTRODUCTION

Dimensional reduction techniques are applicable to studying special solutions to partial differential equations particularly in the presence of a group action where invariant solutions are of interest. The invariant solutions may be interpreted as solutions to an associated set of equations on a lower dimensional space of orbits of the group action. However, one may ask if there is no group action, is it still possible to dimensionally–reduce the original system? A positive answer points to the study of the Hermitian–Einstein (HE) equation with respect to special metrics on holomorphic bundles together with some extra data. The purpose of this paper is to outline a construction leading to dimensional reduction of a class of equations which we call the *perturbed Hermitian–Einstein equations* (briefly, the PHE equations) on a Hermitian holomorphic vector bundle $\mathcal{E} \longrightarrow M$ where M is a compact Kähler manifold. We stress that the term perturbed has here a delicate interpretation as will be apparent from the text. In fact, the PHE equations are actually more general than the HE equations because they possess an extra perturbation term. This extra term arises from the fact that in this case, M is the total space of a holomorphic fiber bundle $F \hookrightarrow M \longrightarrow X$, where X is a compact Kähler manifold and the fiber F is a compact Kählerian homogeneous space. Now $\mathcal{E}$ as a holomorphic vector bundle is obtained via a holomorphic extension of certain holomorphic vector bundles on M and is equipped with an invariant hermitian metric. This metric together with the Kobayashi form of the extension and some natural conditions on F, imply that the PHE equation is equivalent to a system of equations on X, namely the *twisted coupled vortex equations*.

The overall construction, on which there are several variations, relies on results relating to the representation theory of complex semisimple Lie groups and the Bott–Borel–Weil theorem. In addition, the PHE equation can be obtained as a moment map equation. Here we will outline the general construction of [13] leading to the twisted coupled vortex equations (cf [10] [11] [16]). The existence theory of the solutions of such twisted coupled equations is discussed via the Hitchin–Kobayashi correspondence in [12]. References [1] [2] contain an independent study of several aspects of this theory and focus on other questions.

2. Some preliminaries

2.1. **The Kähler manifold** M**.** Let us commence by describing the compact homogeneous Kähler manifold F, that is, for connected complex Lie groups G and P with G semisimple and $P \subset G$ parabolic, we set $F = G/P \cong U/K$ where

$$(2.1.1) \qquad\qquad G = \mathrm{Hol}(F)_e \ , \ U = \mathrm{Hol}_{\mathrm{Iso}}(F)_e \ , \ K = U \cap P \ .$$

Furthermore, F is a simply connected algebraic manifold, the groups U and K are connected compact Lie groups, with U semisimple and K the centralizer of a torus (hence $K \subset U$ has maximal rank) and any G–invariant hermitian metric on F is a Kähler (for further details see [4] [6] [21]). The equivariant holomorphic vector bundles on G/P are homogeneous vector bundles [6] given by representations (ρ, V_ρ) of the parabolic subgroup P

$$(2.1.2) \qquad\qquad \rho \mapsto \mathcal{V}_\rho = G \times_P V_\rho \ .$$

Let X be a compact Kähler manifold and $P_G \to X$ a holomorphic principal G–bundle. The homogeneous vector bundle $\mathcal{V}_\rho$ extends to a holomorphic vector bundle on the associated holomorphic fiber bundle $M = P_G \times_G F = P_G/P$ by the formula

$$(2.1.3) \qquad\qquad \widetilde{\mathcal{V}}_\rho \cong P_G \times_P V_\rho \to M = P_G/P \ .$$

We call $\widetilde{\mathcal{V}}_\rho$ the *canonical extension* of $\mathcal{V}_\rho$. With regards to the fundamental group $\Gamma = \pi_1(X)$, we suppose that M has the structure of a generalized flat bundle [20]

$$(2.1.4) \qquad\qquad F \hookrightarrow M = \widetilde{X} \times_\Gamma F \xrightarrow{\pi} X \ ,$$

with holonomy $\alpha : \Gamma \to U$. Letting ω_F and ω_X denote the Kähler forms of F and X respectively, the extension to M of the (invariant) Kähler form ω_F , is given by $\widetilde{\omega}_F = p^*\omega_F/\alpha$, where $p : \widetilde{X} \times F \to F$, is the natural projection. Then by [11] (Proposition 8.1), there exists a family of Kähler metrics on M with corresponding weighted Kähler forms

$$(2.1.5) \qquad\qquad \omega_\sigma = \pi^*\omega_X + \sigma\widetilde{\omega}_F \ ,$$

where $\sigma > 0$ is a constant parameter.

2.2. **The bundle types of the extension on** M**.** Let $\mathcal{V}_{\rho_i} = U \times_K V_{\rho_i} \to F = U/K$ be homogeneous holomorphic vector bundles with canonical extensions $\widetilde{\mathcal{V}}_{\rho_i} \to M$ for $i = 1, 2$. Further, let $\mathcal{W}_i \to X$ be holomorphic vector bundles and set $\mathcal{E}_i = \pi^*\mathcal{W}_i \otimes_\mathbb{C} \widetilde{\mathcal{V}}_{\rho_i}$. We consider the class of holomorphic vector bundles $\mathcal{E} \to M$ given by proper holomorphic extensions of the form

$$(2.2.1) \qquad\qquad \mathbb{E} \ : \ 0 \to \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \to 0 \ .$$

Such extensions are classified by the Ext^1–functor (see e.g. [18]) which in our case is of the form

$$(2.2.2) \qquad \mathrm{Ext}^1_{\mathcal{O}_M}(\mathcal{E}_2, \mathcal{E}_1) \cong H^{0,1}(M, \mathcal{H}om_\mathbb{C}(\mathcal{E}_2, \mathcal{E}_1)) \cong H^{0,1}(M, \pi^*\mathcal{W} \otimes_\mathbb{C} \widetilde{\mathcal{V}}_\rho) \ ,$$

where we set $\mathcal{W} = \mathcal{H}om_\mathbb{C}(\mathcal{W}_2, \mathcal{W}_1)$ and $\mathcal{V}_\rho = \mathcal{H}om_\mathbb{C}(\mathcal{V}_{\rho_2}, \mathcal{V}_{\rho_1})$. Note that in the latter case we have $\rho = \rho_1 \otimes \rho_2^*$.

For any holomorphic vector bundle $\mathcal{W} \to X$ and any homogeneous vector bundle $\mathcal{V} \to F$, there is an exact sequence derived from the Borel–Leray spectral sequence [11] [19] :

$$(2.2.3) \qquad \begin{aligned} 0 \to H^{0,1}(X, \mathcal{W} \otimes_\mathbb{C} \mathcal{H}^0(F, \mathcal{V})) &\xrightarrow{\pi^*} H^{0,1}(M, \pi^*\mathcal{W} \otimes_\mathbb{C} \widetilde{\mathcal{V}}) \xrightarrow{\Phi} H^0(X, \mathcal{W} \otimes_\mathbb{C} \mathcal{H}^{0,1}(F, \mathcal{V})) \to \\ &\xrightarrow{d_2} H^{0,2}(X, \mathcal{W} \otimes_\mathbb{C} \mathcal{H}^0(F, \mathcal{V})) \xrightarrow{\pi^*} H^{0,2}(M, \pi^*\mathcal{W} \otimes_\mathbb{C} \widetilde{\mathcal{V}}) \ , \end{aligned}$$

where π^* , Φ are the edge homomorphisms.

In the flat case we can say more about the edge map Φ [13]. Here we use the notation $\mathbf{H}^q(F, \mathcal{V})$ to indicate the fact that the fiber cohomologies $\mathcal{H}^q(F, \mathcal{V})$ are flat holomorphic bundles.

Proposition 2.2.1. *Suppose that the fiber bundle $F \hookrightarrow M \to X$ is flat, with holonomy $\alpha : \Gamma \to U$. For any holomorphic vector bundle $\mathcal{W} \to X$ and any equivariant vector bundle $\mathcal{V} \to F$, we have a short exact sequence*

(2.2.4)
$$0 \to H^{0,1}(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^0(F, \mathcal{V})) \xrightarrow{\pi^*} H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}})$$
$$\xrightarrow{\Phi} H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^{0,1}(F, \mathcal{V})) \to 0 .$$

This has the following consequence (cf [11] Proposition 7.1).

Corollary 2.2.2. *Suppose that $\mathcal{V}$ satisfies the vanishing condition*
$$H^0(F, \mathcal{V}) = 0 .$$

Then the following hold :

(1) *The holomorphic extensions of the form (2.2.1) are classified by*
$$\operatorname{Ext}^1_{\mathcal{O}_M}(\mathcal{E}_2, \mathcal{E}_1) \cong H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) \overset{\Phi}{\cong} H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^{0,1}(F, \mathcal{V})) ,$$
   and we have
$$H^0(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) = 0 ,$$
   for any holomorphic vector bundle $\mathcal{W}$ on X .

(2) *If $\Gamma = \pi_1(X)$ acts trivially on $H^{0,1}(F, \mathcal{V})$, then the bundle $\mathcal{H}^{0,1}(F, \mathcal{V})$ of fiber cohomologies is holomorphically trivial and we have the Kunneth formula*
$$H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) \overset{\Phi}{\cong} H^0(X, \mathcal{W}) \otimes_{\mathbb{C}} H^{0,1}(F, \mathcal{V}) .$$

2.3. **The Kobayashi form of the extension.** In order to describe the representative of the extension class one needs to construct a right inverse to the edge homomorphism Φ in Proposition 2.2.1. This is done in [13] and we summarize the necessary results in the proposition below. It will be useful to keep in mind the following diagram of holomorphic maps

(2.3.1)
$$
\begin{array}{ccc}
\widetilde{X} \times F & \xrightarrow{\tilde{\pi}} & \widetilde{X} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_0} \\
M = \widetilde{X} \times_{\Gamma} F & \xrightarrow{\pi} & X
\end{array}
$$

and the relevant cohomology groups as determined by the diagram

(2.3.2)
$$
\begin{array}{ccc}
H^{0,1}(\widetilde{X} \times F, \tilde{q}^*\mathcal{W} \otimes_{\mathbb{C}} p^*\mathcal{V}_\rho)^{\Gamma} & \xrightarrow{\widetilde{\Phi}} & \operatorname{Hom}_{\Gamma}(H^{0,1}(F, \mathcal{V}_\rho)^*, H^0(\widetilde{X}, q_0^*\mathcal{W})) \\
\cong \uparrow{\scriptstyle q^*} & & \cong \uparrow{\scriptstyle q_0^*} \\
H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_\rho) & \xrightarrow{\Phi} & H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^{0,1}(F, \mathcal{V}_\rho)) .
\end{array}
$$

Proposition 2.3.1. *With regards to the edge homomorphism Φ in (2.2.4), we have the following :*

(1) *For a given $\beta_0 \in H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^{0,1}(F, \mathcal{V}_\rho))$ there exists a canonical class $[\bar{\beta}] \in H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_\rho)$ such that $\Phi([\bar{\beta}]) = \beta_0$.*

(2) *The Kobayashi form $\beta \in H^{0,1}(M, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}_2, \mathcal{E}_1)) \cong H^{0,1}(M, \pi^*\mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_\rho)$ of the holomorphic extension (2.2.1)*
$$\mathbb{E} \; : \; 0 \to \pi^*\mathcal{W}_1 \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_{\rho_1} \longrightarrow \mathcal{E} \longrightarrow \pi^*\mathcal{W}_2 \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_{\rho_2} \to 0 ,$$
   decomposes as
$$[\beta] = \pi^*[\beta_X] + [\bar{\beta}] ,$$
   where $[\beta_X] \in H^{0,1}(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^0(F, \mathcal{V}_\rho))$ and $\bar{\beta}$ is the right inverse of β_0 . Thus we have $\Phi([\beta]) = \Phi([\bar{\beta}]) = \beta_0$.

Our discussion of extension classes on M leads naturally to the following definitions of holomorphic objects on X :

- A *holomorphic quadruple* $Q = (\mathcal{W}_1, \mathcal{W}_2, [\beta_X], \beta_0)$ is given by two holomorphic vector bundles $\mathcal{W}_i \to X$, together with cohomology classes

$$[\beta_X] \in H^{0,1}(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^0(F, \mathcal{V}_\rho)) \ , \quad \beta_0 \in H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^{0,1}(F, \mathcal{V}_\rho)) \ .$$

  A holomorphic quadruple of the form $Q = (\mathcal{W}_1, \mathcal{W}_2, 0, 0)$, that is $[\beta_X] = 0$, $\beta_0 = 0$ is called *degenerate* (see [13]).

- A *twisted holomorphic triple* $T_0 = (\mathcal{W}_1, \mathcal{W}_2, \beta_0)$ is given by two holomorphic vector bundles $\mathcal{W}_i \to X$, together with a holomorphic homomorphism

$$\beta_0 \ : \ \mathbf{H}^{0,1}(F, \mathcal{V}_\rho)^* \longrightarrow \mathcal{W} = \mathcal{H}om_{\mathbb{C}}(\mathcal{W}_2, \mathcal{W}_1) \ .$$

  If a basis $\{\hat{\eta}_j\}$ of $H^{0,1}(F, \mathcal{V}_\rho)$ is specified, we denote by T_0 also the k–triple $T_0 = (\mathcal{W}_1, \mathcal{W}_2, \tilde{\phi})$, where $\tilde{\phi} = (\tilde{\phi}_j)_{j=1,\dots,k}$ are the coefficients in the expansion of β_0 (holomorphic triples are considered in [8] [12]).

- A *twisted 1–cohomology triple* $T_1 = (\mathcal{W}_1, \mathcal{W}_2, [\beta_X])$ is given by two holomorphic vector bundles $\mathcal{W}_i \to X$, together with a cohomology class

$$[\beta_X] \in H^{0,1}(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^0(F, \mathcal{V}_\rho)) \ ,$$

  classifying a holomorphic extension on X of the form

(2.3.3) $$\mathbb{W} \ : \ 0 \to \mathcal{W}_1 \otimes_{\mathbb{C}} \mathbf{H}^0(F, \mathcal{V}_\rho) \longrightarrow \widetilde{\mathcal{W}} \longrightarrow \mathcal{W}_2 \to 0$$

  (1–cohomology triples are considered in [9] [14]).

Each of the above classes plays a significant role in the dimensional reduction theory [13]. Here we will restrict attention mainly to holomorphic triples and proceed to state a result which makes use of Corollary 2.2.2 and provides the explicit form of the extension class β .

Lemma 2.3.2. [13] *Suppose that the homogeneous vector bundle $\mathcal{V}_\rho$ satisfies the vanishing condition in Corollary 2.2.2.*

(1) *Relative to a basis $\hat{\eta}_j = [\eta_j]$ of $H^{0,1}(F, \mathcal{V}_\rho)$, the holomorphic triples $T_0 = (\mathcal{W}_1, \mathcal{W}_2, \beta_0)$ are of the form*

$$q_0^* \, \beta_0 = \sum_{j=1}^{k} \ \tilde{\phi}_j \otimes \hat{\eta}_j \ ,$$

where $\tilde{\phi} = (\tilde{\phi}_j)_{j=1,\dots,k} \in H^0(\widetilde{X}, \ q_0^* \mathcal{W})^k$ is a k–tuple of holomorphic sections.

(2) *There is a one–one–correspondence between k–tuples $\tilde{\phi} = (\tilde{\phi}_j)_{j=1,\dots,k}$ of holomorphic sections and extension classes*

$$[\beta] \in \mathrm{Ext}^1_{\mathcal{O}_M}(\mathcal{E}_2, \mathcal{E}_1) \cong H^{0,1}(M, \pi^* \mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}) \cong H^0(X, \mathcal{W} \otimes_{\mathbb{C}} \mathbf{H}^{0,1}(F, \mathcal{V}_\rho)) \ ,$$

given by

$$q^* \beta = \sum_{j=1}^{k} \ \tilde{\pi}^* \tilde{\phi}_j \otimes p^* \eta_j \ .$$

3. The perturbation terms associated to a holomorphic extension

So far we have described how extensions $\mathbb{E}$ in (2.2.1) are classified by $[\beta] \in H^{0,1}(M, \pi^* \mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_\rho)$ and thanks to Lemma 2.3.2 we have an explicit form of β which will be instrumental in the reduction procedure. Owing to the generality of our construction, certain technical features which did not arise in [10] [11] now become apparent and lead to the formulation of the PHE equation.

Henceforth we assume some familiarity with the differential geometry of operators on Kähler manifolds (references are [22] [26]). In particular, Λ_σ will denote the operator of contraction with respect to the Kähler form ω_σ in (2.1.5).

3.1. **Integration over the fiber.** We define integration over the fiber in the flat fiber bundle $M \to X$,

$$(3.1.1) \qquad \pi_* = \int_F \text{ Tr } : \ A^0(M, \ \mathcal{E}nd_\mathbb{C}(\pi^*\mathcal{W}) \otimes_\mathbb{C} \mathcal{E}nd_\mathbb{C}(\widetilde{\mathcal{V}}_\rho)) \longrightarrow \text{End}_\mathbb{C}(X, \mathcal{W}) \ ,$$

at the level of Γ–invariant sections

$$\pi_* = \int_F \text{ Tr } : \ A^0(\widetilde{X} \times F, \ \mathcal{E}nd_\mathbb{C}(\tilde{q}^*\mathcal{W}) \otimes_\mathbb{C} \mathcal{E}nd_\mathbb{C}(p^*\mathcal{V}_\rho))^\Gamma \longrightarrow \text{End}_\mathbb{C}(\widetilde{X}, \ q_0^*\mathcal{W})^\Gamma$$

by the formula

$$(3.1.2) \qquad \int_F \text{ Tr } \tilde{\pi}^*\varphi \otimes p^*\psi = \frac{1}{\text{Vol}(F)} \ \tilde{\varphi} \int_F \text{ Tr}(\psi) \ \text{dvol}_F = \frac{1}{\ell! \ \text{Vol}(F)} \ \tilde{\varphi} \int_F \text{ Tr}(\psi) \ \omega_F^\ell \ .$$

This is well–defined, since the volume form $\text{dvol}_F = \frac{\omega_F}{\ell!}$ is U–invariant. Here Tr is induced by the normalized trace on the fiber, that is the trace on the bundle $\mathcal{E}nd_\mathbb{C}(\mathcal{V}_\rho)$. We remark that the flatness of the fiber bundle is not necessary in order to define integration over the fiber.

Observe that the 'basic' terms β_X and β_0 both involve data on the fiber, holomorphic homomorphisms in the case of β_X and holomorphic extensions in the case of β_0 . There are particular curvature terms $\Lambda_\sigma(\beta \wedge \beta^*)$ and $\Lambda_\sigma(\beta^* \wedge \beta)$ which depend on hermitian metrics h_i on $\mathcal{W}_i$ and the fixed invariant hermitian metrics k_i on the homogeneous bundles $\mathcal{V}_{\rho_i}$.

Lemma 3.1.1.

(1) *The endomorphisms* $-\iota \int_F \text{ Tr } \Lambda_\sigma(\beta \wedge \beta^*) \in \text{End}_\mathbb{C}(\mathcal{W}_1)$ *and* $\iota \int_F \text{ Tr } \Lambda_\sigma(\beta^* \wedge \beta) \in \text{End}_\mathbb{C}(\mathcal{W}_2)$ *are non–negative hermitian endomorphisms of* $\mathcal{W}_i$.

(2) *If* $\int_F \text{ Tr } \Lambda_\sigma(\beta \wedge \beta^*) = 0$ *or* $\int_F \text{ Tr } \Lambda_\sigma(\beta^* \wedge \beta) = 0$, *then* $\beta = 0$.

(3) *For* $\beta = \pi^*\beta_X + \bar{\beta}$ *as in Proposition 2.3.1, we have*

$$\Lambda_\sigma(\beta \wedge \beta^*) = \Lambda_\sigma(\pi^*\beta_X \wedge \pi^*\beta_X^*) + \Lambda_\sigma(\bar{\beta} \wedge \bar{\beta}^*) \ ,$$
$$\Lambda_\sigma(\beta^* \wedge \beta) = \Lambda_\sigma(\pi^*\beta_X^* \wedge \pi^*\beta_X) + \Lambda_\sigma(\bar{\beta}^* \wedge \bar{\beta}) \ .$$

Definition 3.1.2. The *perturbation terms* $\mathfrak{d}_i(\beta, \sigma)$ associated to β are defined by :

$$\mathfrak{d}_1(\beta, \sigma) = \Lambda_\sigma(\beta \wedge \beta^*) - \pi^* \int_F \text{ Tr } \Lambda_\sigma(\beta \wedge \beta^*) \otimes \widetilde{\mathbf{I}}_1 \ ,$$

$$\mathfrak{d}_2(\beta, \sigma) = \Lambda_\sigma(\beta^* \wedge \beta) - \pi^* \int_F \text{ Tr } \Lambda_\sigma(\beta^* \wedge \beta) \otimes \widetilde{\mathbf{I}}_2 \ .$$

The following properties are derived directly from the definition :

(1) $\int_F \text{ Tr } \mathfrak{d}_i(\beta, \sigma) = 0$, that is the perturbation terms vanish under integration over the fiber.

(2) For $\beta = \pi^*\beta_X + \bar{\beta}$ as above, we have $\mathfrak{d}_i(\beta, \sigma) = \mathfrak{d}_i(\pi^*\beta_X, \sigma) + \mathfrak{d}_i(\bar{\beta}, \sigma)$, $i = 1, 2$.

3.2. **The linear maps** λ_i. Relative to an orthonormal basis $\{\hat{\eta}_j\}$ of $H^{0,1}(F, \mathcal{V}_\rho)$, we define linear homomorphisms

$$\lambda_i \ : \ \text{End}_\mathbb{C}(H^{0,1}(F, \mathcal{V}_\rho)) \to \text{End}_\mathbb{C}(\mathcal{V}_{\rho_i}) \ ,$$

by the formulas

$$(3.2.1) \qquad \lambda_1(\eta_{ij}) = \frac{1}{\iota} \ \Lambda_F(\eta_i \wedge \eta_j^*) \ , \quad \lambda_2(\eta_{ij}) = \iota \ \Lambda_F(\eta_i^* \wedge \eta_j) \ ,$$

where η_{ij} is the standard basis of $\text{End}_\mathbb{C}(H^{0,1}(F, \mathcal{V})) \cong \mathfrak{gl}(k, \mathbb{C})$. Since $\eta_{ij}^* = \eta_{ji}$ and $\Lambda_F(\eta_i \wedge \eta_j^*)^* = -\Lambda_F(\eta_j \wedge \eta_i^*)$, we have $\lambda_i(\xi)^* = \lambda_i(\xi^*)$.

There are induced maps on sections

$$\lambda_{i*} \ : \ A^0(X, \ \mathcal{E}nd_\mathbb{C}(\mathcal{W}_i) \otimes_\mathbb{C} \mathbf{End}_\mathbb{C}(H^{0,1}(F, \mathcal{V}_\rho))) \longrightarrow A^0(X, \ \mathcal{E}nd_\mathbb{C}(\mathcal{W}_i) \otimes_\mathbb{C} \mathbf{End}_\mathbb{C}(\mathcal{V}_{\rho_i})) \ ,$$

$$\pi^* \ : \ A^0(X, \ \mathcal{E}nd_\mathbb{C}(\mathcal{W}_i) \otimes_\mathbb{C} \mathbf{End}_\mathbb{C}(\mathcal{V}_{\rho_i})) \longrightarrow A^0(M, \ \mathcal{E}nd_\mathbb{C}(\pi^*\mathcal{W}_i) \otimes_\mathbb{C} \mathcal{E}nd_\mathbb{C}(\widetilde{\mathcal{V}}_{\rho_i})) \ .$$

The main advantage of the maps λ_i consists in the fact that they allow us to express forms like $\Lambda_\sigma(\bar{\beta} \wedge \bar{\beta}^*)$ in terms of basic data.

Lemma 3.2.1. *The forms $\Lambda_\sigma(\bar{\beta} \wedge \bar{\beta}^*)$ and $\Lambda_\sigma(\bar{\beta}^* \wedge \bar{\beta})$ are determined by the formulas*

$$(3.2.2) \qquad \begin{aligned} \Lambda_\sigma(\bar{\beta} \wedge \bar{\beta}^*) &= \frac{\iota}{\sigma} \, \pi^* \, \lambda_{1*}(\beta_0 \wedge \beta_0^*) \ , \\ \Lambda_\sigma(\bar{\beta}^* \wedge \bar{\beta}) &= - \frac{\iota}{\sigma} \, \pi^* \, \lambda_{2*}(\beta_0^* \wedge \beta_0) \ . \end{aligned}$$

For $\beta = \pi^* \beta_X + \bar{\beta}$, $\beta_0 = \Phi(\bar{\beta})$ the perturbation terms $\mathfrak{d}_i(\beta, \sigma)$ are now seen to be of the form

$$\mathfrak{d}_i(\beta, \sigma) = \pi^* \mathfrak{d}_i(\beta_X) + \frac{1}{\sigma} \, \pi^* \mathfrak{d}_i(\beta_0) \ .$$

Observe in particular that the scaling parameter σ for the fiber metric appears as a parameter in the perturbation terms and this is the reason for the terminology.

4. The perturbed Hermitian–Einstein equation

Recalling the holomorphic vector bundle $\mathcal{E} \to M$ in (2.2.1), our next step is to describe an integrable unitary (metric) connection on $\mathcal{E}$ and compute its curvature. Then we will combine this with the background material so far established and proceed to our intended system of equations.

4.1. **The connection on $\mathcal{E} \to M$ and its curvature.** The holomorphic vector bundle $\mathcal{E}$ admits a smooth decomposition $E = E_1 \oplus E_2$. Relative to this decomposition, we denote by $\mathbf{h}$ an invariant hermitian metric on $\mathcal{E}$ of the form

$$(4.1.1) \qquad \mathbf{h} = \mathbf{h}_1 \oplus \mathbf{h}_2 \ ,$$

where $\mathbf{h}_i = h_i' \otimes \tilde{k}_i$ on $\mathcal{E}_i = \pi^* \mathcal{W}_i \otimes_\mathbb{C} \widetilde{\mathcal{V}}_{\rho_i}$ is given by invariant (*basic*) hermitian metrics h_i' on $\pi^* \mathcal{W}_i$ and the extension $\tilde{k}_i$ of U–invariant Hermitian–Einstein metrics k_i on $\mathcal{V}_{\rho_i}$.

Relative to the smooth decomposition of $\mathcal{E}$ and the hermitian metric $\mathbf{h}$, the unitary integrable connection $\mathbf{A}$ on $(\mathcal{E}, \mathbf{h})$ is given by

$$(4.1.2) \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \beta \\ -\beta^* & \mathbf{A}_2 \end{bmatrix} \ ,$$

where $\mathbf{A}_i$ are the Chern connections of $(\mathcal{E}_i, \mathbf{h}_i)$, and $\beta \in A^{0,1}(M, \mathcal{H}om_\mathbb{C}(\mathcal{E}_2, \mathcal{E}_1))$ is the representative of the extension class in $\mathrm{Ext}^1_{\mathcal{O}_M}(\mathcal{E}_2, \mathcal{E}_1)$ relative to (2.2.1). A routine calculation (cf e.g. [22]) shows that the curvature of $\mathbf{A}$ has the form

$$(4.1.3) \qquad F_\mathbf{h} = F_\mathbf{A} = \begin{bmatrix} F_{\mathbf{h}_1} - \beta \wedge \beta^* & D'\beta \\ -D''\beta^* & F_{\mathbf{h}_2} - \beta^* \wedge \beta \end{bmatrix} \ ,$$

where

$$(4.1.4) \qquad D : A^1(M, \mathcal{H}om_\mathbb{C}(\mathcal{E}_1, \mathcal{E}_2)) \to A^2(M, \mathcal{H}om_\mathbb{C}(\mathcal{E}_1, \mathcal{E}_2)) \ ,$$

is constructed from $\mathbf{A}_1$ and $\mathbf{A}_2$ in the standard way. Further, we let D' and D'' denote the $(1,0)$ and $(0,1)$ components of D respectively, so that $D = D' \oplus D''$.

Now for the integrable unitary connection A_i on $(\mathcal{W}_i, h_i)$, and the Hermitian–Einstein metric connection $\widetilde{A}_i$ on $(\widetilde{\mathcal{V}}_{\rho_i}, \tilde{k}_i)$, we have

$$(4.1.5) \qquad \mathbf{A}_i = \pi^* A_i \otimes \widetilde{\mathbf{I}}_i + \mathbf{I}_i \otimes \widetilde{A}_i \ ,$$

and the corresponding curvature form of type $(1,1)$ can be expressed as

$$(4.1.6) \qquad F_{\mathbf{h}_i} = F_{\pi^* h_i} \otimes \widetilde{\mathbf{I}}_i + \mathbf{I}_i \otimes F_{\tilde{k}_i} \ ,$$

where $\mathbf{I}_i = \pi^* \mathbf{I}_{\mathcal{W}_i}$ and $\widetilde{\mathbf{I}}_i = \mathbf{I}_{\widetilde{\mathcal{V}}_{\rho_i}}$.

Next, we define the slope of $\mathcal{E}$ relative to ω_σ for $\sigma > 0$, by

$$(4.1.7) \qquad \lambda = \mu_\mathbb{E}(\sigma) = \mu_\mathcal{E}(\sigma) = \frac{\deg_\sigma(\mathcal{E})}{\operatorname{rank}(\mathcal{E})} \ .$$

We recall that for a compact Kähler manifold (M, ω_M) and a holomorphic vector bundle $\mathcal{E} \to M$, the bundle $\mathcal{E}$ is *stable* (*semistable*) with respect to ω_M, if for any proper holomorphic subbundle $\mathcal{E}' \subset \mathcal{E}$ for which $0 < \operatorname{rank}(\mathcal{E}') < \operatorname{rank}(\mathcal{E})$, we have $\mu_{\mathcal{E}'} < \mu_\mathcal{E}$ (respectively, $\mu_{\mathcal{E}'} \leq \mu_\mathcal{E}$). The bundle $\mathcal{E}$ is said to be *polystable* if $\mathcal{E}$ is a direct sum of stable bundles of equal slope (see e.g. [22]).

Definition 4.1.1. Relative to (M, ω_σ) and $\mathcal{E} \to M$ as above, the *perturbed Hermitian–Einstein equation* (the PHE equation) is defined to be

$$(4.1.8) \qquad \iota \, (\Lambda_\sigma F_\mathbf{h} + \mathfrak{d}(\beta, \sigma)) = 2\pi \, \lambda \, \mathbf{I}_\mathcal{E} \ .$$

If $\mathfrak{d}(\beta, \sigma) = 0$, then (4.1.8) reduces to the usual Hermitian–Einstein equation [22] :

$$\iota \, \Lambda_\sigma F_\mathbf{h} = 2\pi \, \lambda \, \mathbf{I}_\mathcal{E} \ .$$

Remark 4.1.2. The term 'perturbed' signifies the inclusion of the perturbation term $\mathfrak{d}(\beta, \sigma)$ in (4.1.8) which to some extent accounts for the fact that the usual Hermitian–Einstein equation does not in general admit a nice decomposition with respect to the holomorphic fiber bundle $M \to X$, even in the product case. The PHE equation necessitates working with a corresponding restricted type of (poly)stability.

5. A DIMENSIONAL REDUCTION THEOREM

5.1. **The calibration condition on the fiber** F**.** We now fix the fiber data as follows. Recall that on F we are given homogeneous holomorphic bundles $\mathcal{V}_{\rho_i} \to F$ associated to complex representations $(\rho_i \, , V_{\rho_i}) \in R(K)$ as in (2.1.2). The degree $\deg_F(\mathcal{V}_{\rho_i})$ and hence the slope $\mu_{\rho_i} = \mu_{\mathcal{V}_{\rho_i}}$ may be computed in terms of the weights of the representations $(\rho_i \, , V_{\rho_i}) \in R(K)$ by the methods of [5]. Further, the representations $(\rho_i \, , V_{\rho_i}) \in R(K)$ are assumed to be sums of irreducible representations of equal slope. By [22] IV, Prop. 6.1, the K–invariant hermitian metrics on the irreducible components of the representation spaces V_{ρ_i} are unique up to homothety and determine U–invariant Hermitian–Einstein structures on the corresponding homogeneous bundles. It follows that the homogeneous bundles $\mathcal{V}_{\rho_i} \to F$ have U–invariant Hermitian–Einstein structures and are therefore polystable. If the representations $(\rho_i \, , V_{\rho_i})$ are irreducible, then the bundles $\mathcal{V}_{\rho_i}$ are stable (simple) by [22] IV, Prop. 6.4 (cf also [21] [25]). Conversely, if the bundles $\mathcal{V}_{\rho_i}$ are stable, the representations $(\rho_i \, , V_{\rho_i})$ must be irreducible. In this case, two homogeneous bundles $\mathcal{V}_{\rho_i}$ are isomorphic as holomorphic vector bundles, if and only if the representations (ρ_i, V_{ρ_i}) are equivalent [25].

As a consequence, the canonical extensions $\widetilde{\mathcal{V}}_{\rho_i} \to M$ of the homogeneous bundles $\mathcal{V}_{\rho_i} \to F$ admit Hermitian–Einstein structures

$$(5.1.1) \qquad \iota \, \Lambda_\sigma F_{\tilde{k}_i} = 2\pi \, \tilde{\mu}_{\rho_i} \, \widetilde{\mathbf{I}}_i \ ,$$

with constant given by

$$(5.1.2) \qquad \tilde{\mu}_{\rho_i} = \mu_{\widetilde{\mathcal{V}}_i} = \frac{\mu_{\rho_i}}{\sigma} \ .$$

It follows that if $\mathcal{V}_{\rho_i} \to F$ is stable (simple), then $\widetilde{\mathcal{V}}_{\rho_i} \to M$ is stable (simple).

It is reasonable to require certain *calibration conditions* for the homogeneous vector bundles $\mathcal{V}_{\rho_i}$. In view of the Bott–Borel–Weil theorem [6], the representation theory suggests several possibilities. Here we will assume a slope condition of the form

$$(5.1.3) \qquad \mu_\rho = \mu_{\rho_1} - \mu_{\rho_2} < 0 \ .$$

It follows that $H^0(F, \mathcal{V}_\rho) = H^0(F, \mathcal{H}om_\mathbb{C}(\mathcal{V}_{\rho_2}, \mathcal{V}_{\rho_1})) = 0$ and $V_\rho^K = \operatorname{Hom}_K(V_{\rho_2}, V_{\rho_1}) = 0$. Therefore we are in the situation of Corollary 2.2.2. Since the cohomology of the irreducible components must now occur in positive degrees, the Bott–Borel–Weil theorem implies that the corresponding dominant weights must have Bott index ≥ 1 .

Remark 5.1.1. In fact, there are other possible calibration conditions which could be assumed for other choices of the Bott index and consequently a different system of equations on X can be realized [13].

5.2. The reduction of the PHE equation to the twisted coupled vortex equations.

Theorem 5.2.1. *Let $F \hookrightarrow M = \widetilde{X} \times_\Gamma F \xrightarrow{\pi} X$ be a flat holomorphic fiber bundle of compact Kähler manifolds.*

Suppose that the homogeneous holomorphic bundles $\mathcal{V}_{\rho_i}$ on F satisfy the calibration condition (5.1.3) , that is $\mu_\rho = \mu_{\rho_1} - \mu_{\rho_2} < 0$, and therefore $H^0(F, \mathcal{V}_\rho) = 0$.

Consider the proper holomorphic extension

$$(5.2.1) \qquad \mathbb{E} \ : \ 0 \to \pi^*\mathcal{W}_1 \otimes_\mathbb{C} \widetilde{\mathcal{V}}_{\rho_1} \xrightarrow{\ i\ } \mathcal{E} \xrightarrow{\ p\ } \pi^*\mathcal{W}_2 \otimes_\mathbb{C} \widetilde{\mathcal{V}}_{\rho_2} \to 0$$

where $\mathbb{E}$ corresponds to the holomorphic triple $T_0 = (\mathcal{W}_1, \mathcal{W}_2, \beta_0)$.

For $\sigma > 0$, let

$$\lambda = \mu_\mathbb{E}(\sigma) = \mu_\mathcal{E}(\sigma) = \frac{\deg_\sigma(\mathcal{E})}{\mathrm{rank}(\mathcal{E})} \ ,$$

and define the vortex parameters τ_i by

$$\tau_i = \tau_i(\sigma) = \mu_\mathcal{E}(\sigma) - \frac{\mu_{\rho_i}}{\sigma} \ .$$

Then the following statements are equivalent :

(1) *There exist invariant hermitian metrics of the form $\mathbf{h}$ on the extension bundle $\mathcal{E}$ which satisfy the perturbed Hermitian–Einstein equation*

$$(5.2.2) \qquad \iota \ (\Lambda_\sigma F_\mathbf{h} + \frac{1}{\sigma} \ \pi^*\mathfrak{d}(\beta_0)) = 2\pi \ \lambda \ \mathbf{I}_\mathcal{E} \ ,$$

   relative to (M, ω_σ) .

(2) *There exist hermitian metrics h_i on $\mathcal{W}_i$ which satisfy the twisted coupled vortex equations*

$$(5.2.3) \qquad \begin{aligned} \iota \ \Lambda_X F_{h_1} + \ \frac{1}{\sigma} \ \int_F \ \mathrm{Tr} \ \lambda_{1*}(\beta_0 \wedge \beta_0^*) &= 2\pi \ \tau_1 \ \mathbf{I}_{\mathcal{W}_1} \ , \\ \iota \ \Lambda_X F_{h_2} - \ \frac{1}{\sigma} \ \int_F \ \mathrm{Tr} \ \lambda_{2*}(\beta_0^* \wedge \beta_0) &= 2\pi \ \tau_2 \ \mathbf{I}_{\mathcal{W}_2} \ . \end{aligned}$$

(3) *There exist hermitian metrics h_i on $\mathcal{W}_i$ which satisfy the twisted coupled multivortex equations*

$$(5.2.4) \qquad \begin{aligned} \iota \ \Lambda_X F_{h_1} + \frac{1}{\sigma} \ \Phi_1 &= 2\pi \ \tau_1 \ \mathbf{I}_{\mathcal{W}_1} \ , \\ \iota \ \Lambda_X F_{h_2} - \frac{1}{\sigma} \ \Phi_2 &= 2\pi \ \tau_2 \ \mathbf{I}_{\mathcal{W}_2} \ , \end{aligned}$$

   where $\Phi_i \in \mathrm{End}_\mathbb{C}(\mathcal{W}_i)$ are non–negative hermitian endomorphisms satisfying

$$q_0^*\Phi_1 = \sum_{j=1}^k \ \tilde{\phi}_j \circ \tilde{\phi}_j^* \ , \quad q_0^*\Phi_2 = \sum_{j=1}^k \ \tilde{\phi}_j^* \circ \tilde{\phi}_j \ .$$

   Here $k = \dim_\mathbb{C} H^{0,1}(F, \mathcal{V}_\rho)$, and the adjoints ϕ_j^* are taken with respect to the metrics h_i .

There is a one–to–one correspondence between solutions in (1) and (2), (3) given by the assignment $h_i \mapsto h_i' = \pi^* h_i$.

Remark 5.2.2. The data in the equations (5.2.3) and (5.2.4) depend only on the associated holomorphic triple $T_0 = (\mathcal{W}_1, \mathcal{W}_2, \beta_0)$.

Corollary 5.2.3. *If the extension $\mathbb{E}$ is holomorphically split, that is $[\beta] = 0$, the solutions of the PHE on $\mathcal{E}$ relative to (M, ω_σ) , respectively the corresponding solutions (h_1, h_2) of the twisted coupled vortex equations (5.2.4), degenerate to solutions of the uncoupled Hermitian–Einstein equations on each $\mathcal{W}_i$.*

5.3. **Outline of the proof.** The proof of the theorem follows from some technical lemmas which reflect in part upon the flat structure of (2.1.4). We will outline several of the steps involved following [13] extending the special cases of [11] [17].

First of all, we have :

(1) $\Lambda_\sigma F_{\pi^* h_i} = \pi^* \Lambda_X F_{h_i}$;

(2) $\Lambda_\sigma F_{\tilde{k}_i} = \frac{1}{\sigma} \widetilde{\Lambda_F F_{k_i}}$.

Next, using (4.1.3), (4.1.6) and (5.1.1), the PHE equation is equivalent to the equation

$$(5.3.1) \quad \begin{bmatrix} \pi^*(\iota \, \Lambda_X F_{h_1} + 2\pi \, (\tilde{\mu}_{\rho_1} - \lambda) \, \mathbf{I}_{\mathcal{W}_1}) \otimes \widetilde{\mathbf{I}}_{\rho_1} & \iota \, \Lambda_\sigma D' \beta \\ -\iota \, \Lambda_\sigma D'' \beta^* & \pi^*(\iota \, \Lambda_X F_{h_2} + 2\pi \, (\tilde{\mu}_{\rho_2} - \lambda) \, \mathbf{I}_{\mathcal{W}_2}) \otimes \widetilde{\mathbf{I}}_{\rho_2} \end{bmatrix}$$
$$= \iota \begin{bmatrix} \Lambda_\sigma(\beta \wedge \beta^*) - \mathfrak{d}_1(\beta, \sigma) & 0 \\ 0 & \Lambda_\sigma(\beta^* \wedge \beta) - \mathfrak{d}_2(\beta, \sigma) \end{bmatrix} .$$

By the definition of the perturbation terms, this last expression equals

$$\iota \begin{bmatrix} \pi^* \int_F \text{Tr} \, \Lambda_\sigma(\beta \wedge \beta^*) \otimes \widetilde{\mathbf{I}}_{\rho_1} & 0 \\ 0 & \pi^* \int_F \text{Tr} \, \Lambda_\sigma(\beta^* \wedge \beta) \otimes \widetilde{\mathbf{I}}_{\rho_2} \end{bmatrix} .$$

Hence we obtain the equivalent system of equations :

$$(5.3.2) \qquad \Lambda_\sigma D' \beta = 0 \quad , \quad \Lambda_\sigma D'' \beta^* = 0 ,$$

and

$$(5.3.3) \qquad \iota \, \Lambda_X F_{h_1} - 2\pi \, \tau_1(\sigma) \, \mathbf{I}_{\mathcal{W}_1} = \iota \int_F \text{Tr} \, \Lambda_\sigma(\beta \wedge \beta^*) ,$$
$$\iota \, \Lambda_X F_{h_2} - 2\pi \, \tau_2(\sigma) \, \mathbf{I}_{\mathcal{W}_2} = \iota \int_F \text{Tr} \, \Lambda_\sigma(\beta^* \wedge \beta) .$$

The remainder of the proof deals with some analysis of β and showing that the off–diagonal terms in (5.3.1) are zero. It follows from Lemma 2.3.2 that we may choose the smooth decomposition of $\mathcal{E}$, such that $q^* \beta$ is of the form

$$q^* \beta = \sum_{j=1}^k \tilde{\pi}^* \tilde{\phi}_j \otimes p^* \eta_j ,$$

where $\eta_j \in A^{0,1}(F, \mathcal{V}_\rho)$ are $\Delta_{\bar{\partial}}$–harmonic $(0,1)$–forms representing an orthonormal basis of $H^{0,1}(F, \mathcal{V}_\rho)$. Combining this with the Hodge formulas

$$(5.3.4) \qquad \Lambda_\sigma D' \beta - D' \Lambda_\sigma \beta = \iota \, \bar{\partial}^* \beta ,$$
$$\Lambda_\sigma \bar{\partial} \beta - \bar{\partial} \Lambda_\sigma \beta = - \iota \, D'^* \beta .$$

in [22] , we obtain the following equivalent properties for the metrics h_i on $\mathcal{W}_i$ (cf [10]).

Lemma 5.3.1. *Suppose that $\Delta_{\bar{\partial}} \eta_j = 0$, that is the forms $\eta_j \in A^{0,1}(F, \mathcal{V}_\rho)$ are harmonic. Then we have*

(1) $\Lambda_\sigma D' \beta = 0$;

(2) $\Lambda_\sigma D'' \beta^* = 0$;

(3) $\bar{\partial}^* \beta = 0$;

(4) $\Delta_{\bar{\partial}} \beta = 0$, *that is the form $\beta \in A^{0,1}(M, \pi^* \mathcal{W} \otimes_{\mathbb{C}} \widetilde{\mathcal{V}}_\rho)$ is harmonic.*

From the calibration condition (5.1.3), Corollary 2.2.2 and Lemma 3.2.2 we deduce that

$$(5.3.5) \qquad \frac{1}{\iota} \int_F \text{Tr} \, \Lambda_\sigma(\beta \wedge \beta^*) = \frac{1}{\sigma} \int_F \text{Tr} \, \lambda_{1*}(\beta_0 \wedge \beta_0^*) = \frac{1}{\sigma} \, \Phi_1 ,$$
$$\iota \int_F \text{Tr} \, \Lambda_\sigma(\beta^* \wedge \beta) = \frac{1}{\sigma} \int_F \text{Tr} \, \lambda_{2*}(\beta_0^* \wedge \beta_0) = \frac{1}{\sigma} \, \Phi_2 .$$

Using again the above expression from Lemma 2.3.2, the non–negative hermitian endomorphisms Φ_i of $\mathcal{W}_i$ admit the expansion

$$(5.3.6) \qquad q_0^* \Phi_1 = \sum_{i,j} \tilde{\phi}_i \circ \tilde{\phi}_j^* \langle \eta_i, \eta_j \rangle = \sum_{j=1}^k \tilde{\phi}_j \circ \tilde{\phi}_j^* \ , \quad q_0^* \Phi_2 = \sum_{i,j} \tilde{\phi}_i^* \circ \tilde{\phi}_j \langle \eta_i^*, \eta_j^* \rangle = \sum_{j=1}^k \tilde{\phi}_j^* \circ \tilde{\phi}_j$$

The Theorem now follows essentially from (5.3.3), (5.3.5) and (5.3.6).

5.4. **The Reduction Theorem for invariant extensions.** Here we assume the following stronger conditions on the data on the fiber.

(1) The representations $(\rho_i, V_{\rho_i}) \in R(K)$ are irreducible.

(2) $\mu_\rho = \mu_{\rho_1} - \mu_{\rho_2} < 0$.

(3) $H^{0,1}(F, \mathcal{V}_\rho)^G \neq 0$.

It follows from the Bott–Borel–Weil theorem [6] and the multiplicity formulas in [24] that the multiplicity of the trivial representation in $H^{0,1}(F, \mathcal{V}_\rho)$ is at most 1 , that is we have $H^{0,1}(F, \mathcal{V}_\rho)^G \cong H^1(\mathfrak{p}, \mathfrak{k}_{\mathbb{C}}; V_\rho) \cong \mathbb{C}$. Under the above assumptions, the terms $\mathfrak{d}_i(\beta_0)$ vanish and Theorem 5.2.1 takes on a more familiar form. In fact, we are now essentially in the situation of [11, Theorem 8.9].

Theorem 5.4.1. *Suppose that the extension* $\mathbb{E}$ *is invariant, that is the Kobayashi form* $[\beta]$ *of* $\mathbb{E}$ *satisfies*

$$[\beta] \in \mathrm{Ext}^1_{\mathcal{O}_M}(\mathcal{E}_2, \mathcal{E}_1)_{\mathbf{1}} \cong H^0(X, \mathcal{W}) \otimes_{\mathbb{C}} H^{0,1}(F, \mathcal{V}_\rho)^U \ .$$

Then the following statements are equivalent:

(1) *The invariant metric* $\mathbf{h}$ *satisfies the Hermitian–Einstein equation*

$$\iota \, \Lambda_\sigma F_{\mathbf{h}} = 2\pi \, \lambda \, \mathbf{I}_{\mathcal{E}} \ .$$

(2) *The metrics* h_1 *and* h_2 *satisfy the coupled vortex equations* :

$$(5.4.1) \qquad \begin{aligned} \iota \Lambda_X F_{h_1} + \frac{1}{\sigma} \, \phi \circ \phi^* &= 2\pi \, \tau_1 \, \mathbf{I}_{\mathcal{W}_1} \ , \\ \iota \, \Lambda_X F_{h_2} - \frac{1}{\sigma} \, \phi^* \circ \phi &= 2\pi \, \tau_2 \, \mathbf{I}_{\mathcal{W}_2} \ . \end{aligned}$$

5.5. **Examples.** Bott's generalization of the Borel–Weil theorem [6] states that for an irreducible P–module (ρ, V_ρ) , the induced cohomology $H^{0,*}(G/P, \mathcal{V}_\rho)$ is either equal to zero or it is an irreducible G–module. The theory underlying the Bott–Borel–Weil (BBW) theorem can be used to compute examples of irreducible P–modules $\mathcal{V}_\rho$ which satisfy the calibration conditions for both of the reduction theorems as stated above. In principle it seems that a plentiful supply of such examples can be computed for many types of the Kähler homogeneous space $F = G/P$, in particular for the case of invariant extensions. The reference [4] (Chapters 1–5) outlines a technology for doing this by means of computational rules. It is based on enumerating the theory of affine actions of the Weyl group of $\mathfrak{g}$ and the Bott–Kostant induction (cf [13]).

The procedure starts by considering the Dynkin diagram for a given $\mathfrak{g}$ where one or more nodes $\bullet$ are replaced by a crossed node $\times$ when there is a non–parabolic simple root. In this way the Dynkin diagram for $F = G/P$ is obtained. For instance, a maximal parabolic $\mathfrak{p}$ subalgebra (as in the case of the compact irreducible Hermitian symmetric spaces) admits a single crossed node and a Borel subalgebra $\mathfrak{b}$ has crosses through every node as is the case for the full flag manifold G/B over $\mathbb{C}^{\ell+1}$. The weights of the representations are exhibited by such diagrams (see below) by placing (integer) coefficients over each node in accordance with certain rules. If the aim is to obtain a 1–dimensional irreducible $\mathfrak{g}$–module, we would select suitable node coefficients for the diagram such that on taking a single affine Weyl group reflection over the appropriate crossed node leads to zeros over each node in the diagram and hence this selection corresponds to the trivial $\mathfrak{g}$–module provided by the BBW theorem.

Example 5.5.1. Let $F = \mathbb{C}P^4$ and take $\mathcal{V}_\rho$ to be the irreducible P–module Ω_F^1 . Starting from the corresponding Dynkin diagram $\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$ and then taking a single affine reflection, the trivial module is obtained. Thus $H^1(F, \mathcal{V}_\rho) \cong \mathbb{C}$, and the cohomology in all other degrees is zero by the BBW Theorem.

The dual module $\mathcal{V}_\rho^* \cong \mathcal{T}_F^{1,0}$ has the corresponding diagram $\overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$. That for the canonical line bundle $\mathcal{K}_F = \Omega_F^4$ is $\overset{-5}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ from which $\mathcal{V}_\rho \otimes \mathcal{K}_F$ is represented by $\overset{-7}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$. On taking four affine reflections on the latter we obtain $\overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$. Thus Serre–duality and the BBW theorem imply that the irreducible G–module

$$H^0(F, \mathcal{V}_\rho^*) \cong H^4(F, \mathcal{V}_\rho \otimes \mathcal{K}_F) \cong \mathfrak{g} \; : \; \overset{1}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$$

and the cohomology in all other degrees is zero, so we have $H^0(F, \mathcal{V}_\rho^*)^U = H^1(F, \mathcal{V}_\rho^*)^U = 0$.

Example 5.5.2. Here we take F to be the 9–dimensional partial flag manifold over $\mathbb{C}^5$ whose compact representation is

$$F \cong SU(5)/S(U(1) \times U(2) \times U(1) \times U(1)) .$$

It is an example of a homogeneous Kähler manifold which is not a symmetric space. Consider the irreducible P–module $\mathcal{V}_\rho$ as represented by $\overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\times}$. A single affine reflection leads to $H^1(F, \mathcal{V}_\rho) \cong \mathbb{C}$ and zero cohomology in all other degrees. As for the dual module $\mathcal{V}_\rho^*$, the diagram is $\overset{1}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{0}{\times}$ which can be seen to correspond to a singular weight and hence $H^q(F, \mathcal{V}_\rho^*) = 0$, for all $q \geq 0$.

Remark 5.5.3. We remark that other types of examples can be formulated following e.g. [23] Theorem B.

6. The moment map and the PHE equation

Given a hermitian metric h on E , the space $\mathcal{C}(E)$ of integrable $\bar{\partial}$–operators, for which $\bar{\partial}^2 = 0$, corresponds bijectively to the space $\mathcal{A}(E, h)$ of unitary integrable connections whose curvature satisfies $F_h^{0,2} = 0$. Here E denotes the underlying smooth vector bundle of $\mathcal{E}$. Thus each element $\bar{\partial}_E \in \mathcal{C}(E)$ defines a unique holomorphic structure $\mathcal{E} = (E, \bar{\partial}_E)$ on E , for which it is the canonical $\bar{\partial}$–operator. The complex gauge group $\mathrm{Aut}(E)$ acts on $\mathcal{C}(E)$ via the action $g(\bar{\partial}) = g \circ \bar{\partial} \circ g^{-1}$, for $g \in \mathrm{Aut}(E)$. For our purpose, we restrict attention to the *unitary* gauge (sub)group denoted by $\mathcal{G}$. The quotient $\mathcal{C}(E)/\mathcal{G}$ is the space of equivalence classes of integrable holomorphic structures on E up to unitary equivalence. The Lie algebra of $\mathcal{G}$ is given by $\mathrm{Lie}(\mathcal{G}) \cong \mathrm{End}_s(E)$, where $\mathrm{End}_s(E)$ is the Lie algebra of global skew–hermitian endomorphisms of E . Background references to this section are [3] [15] [22].

6.1. The restricted gauge group and the moment map.

Since M is Kähler, the inner product

$$(6.1.1) \qquad \langle \alpha_1, \alpha_2 \rangle = \frac{1}{\iota(m-1)! \, \mathrm{Vol}(M)} \int_M \mathrm{Tr}\,(\alpha_1 \wedge \alpha_2^*) \wedge \omega_\sigma^{m-1} ,$$

for $\alpha_1, \alpha_2 \in T_{\bar{\partial}}\,\mathcal{C}(E) \cong A^{0,1}(M, \mathrm{End}_s(E))$, induces a Kähler structure on $\mathcal{C}(E)$ where the Kähler form ω is defined by $\omega(\alpha_1, \alpha_2) = \mathrm{im}\,\langle \alpha_1, \alpha_2 \rangle$. The standard action of $\mathcal{G}$ on $\mathcal{C}(E)$ preserves ω and induces an associated equivariant moment map

$$(6.1.2) \qquad \begin{aligned} \nu = \nu(\mathcal{G}) \; : \; \mathcal{C}(E) &\longrightarrow \mathrm{Lie}(\mathcal{G}) \subset \mathrm{Lie}(\mathcal{G})^* \cong L^2(\mathrm{Lie}(\mathcal{G})) \\ \bar{\partial} &\mapsto \Lambda_\sigma F_h , \end{aligned}$$

where $\bar{\partial}$ corresponds to the unitary integrable connection (A, h) . This moment map is determined up to a constant in the center of the Lie algebra and may also be written as

$$(6.1.3) \qquad \nu(\bar{\partial}) = \Lambda_\sigma F_h \; + \; 2\pi\iota \, \lambda \, \mathbf{I}_\mathcal{E} .$$

Note that $\nu^{-1}(0)$ is empty unless $\lambda = \mu_E$, the slope of E .

In this section we assume that the representations (ρ_i, V_{ρ_i}) are irreducible. We consider the subspace $\mathcal{A}(\mathbb{E}, \mathbf{h}) \subset \mathcal{A}(E, \mathbf{h})$ of unitary integrable connections $\mathbf{A}$ of the form $(\mathbf{A}_1, \mathbf{A}_2, \beta)$, as in (4.1.2), where $\mathbf{h}$ is a (fixed) special metric $\mathbf{h}$ as in (4.1.1). Let $\mathcal{C}(\mathbb{E}) \subset \mathcal{C}(E)$ be the subspace of holomorphic structures determined by $\mathcal{A}(\mathbb{E}, \mathbf{h})$. The elements $\bar{\partial}_E \in \mathcal{C}(\mathbb{E})$ determine a holomorphic structure on the extension $\mathbb{E}$ in (5.2.1), that is $\bar{\partial}_E$ is of the form

$$(6.1.4) \qquad \bar{\partial}_E = \begin{bmatrix} \bar{\partial}_{\pi^* W_1} \otimes \widetilde{\mathbf{I}}_1 + \mathbf{I}_1 \otimes \bar{\partial}_{\widetilde{\mathcal{V}}_1} & \beta \\ 0 & \bar{\partial}_{\pi^* W_2} \otimes \widetilde{\mathbf{I}}_2 + \mathbf{I}_2 \otimes \bar{\partial}_{\widetilde{\mathcal{V}}_2} \end{bmatrix} .$$

We further consider the subspaces $\mathcal{C}_0(\mathbb{E}) \cong \mathcal{A}_0(\mathbb{E}, \mathbf{h})$, consisting of the elements in $\mathcal{C}(\mathbb{E}) \cong \mathcal{A}(\mathbb{E}, \mathbf{h})$ such that β is $\Delta_{\bar{\partial}}$–harmonic, that is $\bar{\partial}^* \beta = 0$. Then $\mathcal{A}_0(\mathbb{E}, \mathbf{h})$ admits a mapping

$$(6.1.5) \qquad \mathcal{H}_{\bar{\partial}}^{0,1}(M, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}_2, \mathcal{E}_1)) \longrightarrow \mathcal{A}_0(\mathbb{E}, \mathbf{h}) \xrightarrow{\Pi} \mathcal{A}(W_1, h_1) \times \mathcal{A}(W_2, h_2) ,$$

where the dimension $\dim_{\mathbb{C}} \mathcal{H}_{\bar{\partial}}^{0,1}(M, \mathcal{H}om_{\mathbb{C}}(\mathcal{E}_2, \mathcal{E}_1))$ is upper–semicontinuous as a function on the base.

We specify a subgroup $\mathcal{G}_0 \subset \mathcal{G}$ which acts symplectically on $\mathcal{C}(E)$ and on $\mathcal{C}_0(\mathbb{E})$ via restriction to the latter. For $u_i \in \mathcal{G}_{W_i}$, the subgroup $\mathcal{G}_0$ is defined by

$$(6.1.6) \qquad \mathcal{G}_0 = \left\{ \begin{bmatrix} \pi^* u_1 \otimes \widetilde{\mathbf{I}}_1 & 0 \\ 0 & \pi^* u_2 \otimes \widetilde{\mathbf{I}}_2 \end{bmatrix} \right\} \cong \mathcal{G}_{W_1} \times \mathcal{G}_{W_2} \subset \mathcal{G} .$$

The subgroup $\mathcal{G}_0$ leaves $\mathcal{C}_0(\mathbb{E})$ invariant and fixes the holomorphic structures on the fiber. In fact, $\mathcal{G}_0$ is the maximal subgroup of $\mathcal{G}$ with this property, since by the irreducibility of the V_{ρ_i} there are no non–trivial U–equivariant gauge transformations on the homogeneous bundles $\mathcal{V}_{\rho_i}$, that is $\mathrm{End}_U(\mathcal{V}_{\rho_i}) \cong \mathrm{End}_{\mathbb{C}}(V_{\rho_i})^K \cong \mathbb{C} \cdot \mathrm{Id}$.

On smooth elements (relative to $\mathbf{h} = \mathbf{h}_1 \oplus \mathbf{h}_2$ as above), we have a commutative diagram

$$(6.1.7) \qquad \begin{array}{ccc} \mathrm{Lie}(\mathcal{G}) & \xrightarrow{\subset} & \mathrm{Lie}(\mathcal{G})^* \cong L^2(\mathrm{Lie}(\mathcal{G})) \\ \uparrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle P_0} \\ \mathrm{Lie}(\mathcal{G}_0) & \xrightarrow{\subset} & \mathrm{Lie}(\mathcal{G}_0)^* \cong L^2(\mathrm{Lie}(\mathcal{G}_0)) \end{array}$$

where P_0 denotes orthogonal projection and for $a \in \mathrm{Lie}(\mathcal{G}_0)$ the following relationship is satisfied :

$$(6.1.8) \qquad \langle P_0(\Lambda_\sigma F_{\mathbf{h}}) , a \rangle = \frac{\iota}{n! \,\mathrm{Vol}(X)} \int_X \mathrm{Tr} \, (P_0(\Lambda_\sigma F_{\mathbf{h}}) \circ a^*) \, \omega_X^n .$$

Observing that the projection P_0 is essentially given by integration over the fiber, we obtain the main result concerning the moment map interpretation of the PHE equation.

Theorem 6.1.1. [13] *With regards to the inclusion $j : \mathcal{C}_0(\mathbb{E}) \hookrightarrow \mathcal{C}(E)$, consider the map*

$$\nu_0 : \mathcal{C}_0(\mathbb{E}) \longrightarrow \mathrm{Lie}(\mathcal{G}_0)^* ,$$

as defined by $\nu_0 = P_0 \circ \nu \circ j$. Then the following hold :

(1) *The map ν_0 is a moment map for the action of $\mathcal{G}_0$ on $\mathcal{C}_0(\mathbb{E})$.*

(2) *The following diagram commutes with respect to the inclusion of smooth elements :*

$$\begin{array}{ccccc} \mathcal{C}(E) & \xrightarrow{\nu} & \mathrm{Lie}(\mathcal{G}) & \xrightarrow{\subset} & \mathrm{Lie}(\mathcal{G})^* \cong L^2(\mathrm{Lie}(\mathcal{G})) \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle P_0} \\ \mathcal{C}_0(\mathbb{E}) & \xrightarrow{\nu_0} & \mathrm{Lie}(\mathcal{G}_0) & \xrightarrow{\subset} & \mathrm{Lie}(\mathcal{G}_0)^* \cong L^2(\mathrm{Lie}(\mathcal{G}_0)) \end{array}$$

(3) *The PHE equation*

$$\iota(\Lambda_\sigma F_{\mathbf{h}} + \mathfrak{d}(\beta, \sigma)) = 2\pi \, \lambda \, \mathbf{I}_{\mathcal{E}} ,$$

is equivalent to the $\mathcal{G}_0$–moment map equation $\nu_0(\bar{\partial}) = 0$.

References

[1] Álvarez–Cónsul, L. and García-Prada, O., *Dimensional reduction, $SL(2, \mathbb{C})$–equivariant bundles and stable holomorphic chains*, preprint École Poly. No. 99–17, 1999.

[2] Álvarez–Cónsul, L. and García-Prada, O., *A Hitchin–Kobayashi correspondence for equivariant holomorphic bundles, quivers and vortices*, in preparation.

[3] Atiyah, M.F. and Bott, R ., *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Royal Soc. London A **308** (1982), 523–615.

[4] Baston, R. and Eastwood, M., *The Penrose Transform: Its Interaction with Representation Theory*, Oxford Math. Monographs, Clarendon Press, Oxford 1989.

[5] Borel, A. and Hirzebruch, F., *Characteristic classes and homogeneous spaces, I, II, III.*, Amer. Jour. Math. **80** (1958), 458–538, **81** (1959), 315–382, **82** (1960), 491–504.

[6] Bott, R., *Homogeneous vector bundles*, Ann. Math. **66** (1957), 203–248.

[7] Bradlow, S.B., Daskalopoulos, G., García-Prada, O. and Wentworth, R.A., *Stable augmented bundles over Riemann surfaces,* in Vector Bundles in Algebraic Geometry (Eds. Hitchin et al.), Cambridge Univ. Press 1995, pp. 15–67.

[8] Bradlow, S.B. and García-Prada, O., *Stable triples, equivariant bundles and dimensional reduction*, Math. Ann. **304** (1995), 225–252.

[9] Bradlow, S.B. and García-Prada, O., *Higher cohomology triples and holomorphic extensions*, Comm. Anal. Geom. **3** (1995), 421–463.

[10] Bradlow, S.B., Glazebrook, J.F. and Kamber, F.W., *A new look at the vortex equations and dimensional reduction*, in Proc. of the First Brazil–USA Workshop on Geometry, Topology and Physics 1996, Verlag Walter de Gruyter & Co, Berlin 1997, pp. 85–106.

[11] Bradlow, S.B., Glazebrook, J.F. and Kamber, F.W., *Reduction of the Hermitian–Einstein equation on Kählerian fiber bundles*, Tohoku Math. Jour. **51** (1999), 81–124.

[12] Bradlow, S.B., Glazebrook, J.F. and Kamber, F.W., *The Hitchin–Kobayashi correspondence for twisted triples*, Int. Jour. of Math. **11**, No. 4 (2000), 493–508.

[13] Bradlow, S.B., Glazebrook, J.F. and Kamber, F.W., *Equivariant dimensional reduction and the Bott–Borel–Weil theorem*, preprint University of Illinois.

[14] Daskalopoulos, G., Uhlenbeck, K. and Wentworth, R., *Moduli of extensions of holomorphic bunles on Kähler manifolds*, Comm. Anal. Geom. **3** No. 3 (1995), 479–522.

[15] Donaldson, S.K., *Anti–self–dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985), 1–26.

[16] García-Prada, O., *Invariant connections and vortices*, Comm. Math. Phys. **156** (1993), 527–546.

[17] García-Prada, O., *Dimensional reduction of stable bundles, vortices and stable pairs,* Int. Jour. Math. **5** (1994), 1–52.

[18] Hartshorne, R., *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer Verlag, Berlin–Heidelberg–New York 1977.

[19] Hirzebruch, F., *Topological Methods in Algebraic Geometry*, Grundlehren der Math. Wiss. **131** (3rd. Ed.), Springer Verlag, Berlin–Heidelberg–New York 1966.

[20] Kamber, F.W. and Tondeur, Ph., *Foliated Bundles and Characteristic Classes*, Lect. Notes in Math., **493**, Springer Verlag, Berlin–Heidelberg–New York, 1975.

[21] Kobayashi, S., *Homogeneous vector bundles and stability*, Nagoya Math. Jour. **101** (1986), 37–54.

[22] Kobayashi, S., *Differential Geometry of Complex Vector Bundles*, Princeton University Press, Princeton 1987.

[23] Merkulov, S. and Schwachhöfer, L., *Classification of irreducible holonomies of torsion–free affine connections*, Ann. Math. **150** (1999), 77–149.

[24] Parthasarathy, K. R., Ranga Rao, R., Varadarajan, V.S., *Representations of complex semi–simple Lie groups and Lie algebras*, Ann. Math. **8**5 (1967), 383–429.

[25] Ramanan, S., *Holomorphic vector bundles on homogeneous spaces*, Topology **5** (1966), 159–177.

[26] Weil, A., *Variétés Kähleriennes*, Act. Sci. et Ind. **1267**, Hermann, Paris 1958.

Department of Mathematics, University of Illinois at Urbana–Champaign, Urbana IL 61801 USA
E-mail address, S. B. Bradlow: `bradlow@math.uiuc.edu`

Department of Mathematics, Eastern Illinois University, Charleston IL 61920 USA, and Department of Mathematics, University of Illinois at Urbana–Champaign, Urbana IL 61801 USA
E-mail address, J. F. Glazebrook: `cfjfg@eiu.edu , glazebro@math.uiuc.edu`

Department of Mathematics, University of Illinois at Urbana–Champaign, Urbana IL 61801 USA
E-mail address, F. W. Kamber: `kamber@math.uiuc.edu`